代数幾何学シンポジウム記録
2007年度　pp.3-5

MORDELL-WEIL GROUP OF AN ABELIAN FIBERED VARIETY AND ITS APPLICATION TO HYPERKÄHLER MANIFOLDS

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We work over $\mathbf{C}$. In 70's, Shioda [Sh] proved the following important:

Theorem 0.1. *Let $f : S \longrightarrow C$ be a relatively minimal Jacobian fibration, i.e., a relatively minimal elliptic fibration with a section O, having at least one singular fibers, say, S_{t_i} $(1 \le i \le k)$. Then, the Mordell-Weil group* $\mathrm{MW}(f)$ *is a finitely generated abelian group of rank*

$$\mathrm{mw}\,(f) = \rho(S) - 2 - \sum_{i=1}^{k}(m_i - 1)\ .$$

Here $\rho(S)$ is the Picard number of S and m_i is the number of irreducible components of S_{t_i}. In particular, $\rho(S) \ge 2$ and $\mathrm{mw}\,(f) \le \rho(S) - 2$.

It is natural to ask the optimality of the last estimate. In this direction, the following result was shown by [O1] (note that $\rho(S) \le 20$ for a K3 surface S):

Theorem 0.2. *Let ρ be an integer s.t. $2 \le \rho \le 20$. Then, for each such ρ, there is a Jacobian K3 surface $f : S \longrightarrow \mathbf{P}^1$ s.t. $\rho(S) = \rho$ and* $\mathrm{mw}\,(f) = \rho - 2$.

In the talk, I explained possible generalizations of these two theorems.

Definition 0.3. Let $f : X \longrightarrow Y$ be a surjective morphism between normal projective varieties. We call f an *abelian fibration* if f has a rational section O and the generic fiber (in the sense of scheme) $A_K := X_\eta$ is an abelian variety defined over $K := \mathbf{C}(Y)$ with origin $O \in A_K(K)$. The *Mordell-Weil group* $\mathrm{MW}\,(f)$ of f is the set of K-rational points $A_K(K)$, or more geometrically, the set of rational sections of f.

$\mathrm{MW}\,(f)$ forms an abelian group and acts faithfully on X as birational automorphisms of X. We assume the following:
(i) X and Y have only $\mathbf{Q}$-factorial rational singularities;
(ii) there is no prime divisor D on X s.t. $f(D)$ is of codimension ≥ 2 on Y;
(iii) $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$.

The conditions (i) and (ii) are natural in the view of flattening theorem and probably the minimal model theory for higher dimensional varieties. Some condition like (iii) is necessary for the finite generation of $MW(f)$. For instance, $MW(p_2)$ is far from being finitely generated for the product manifold $p_2 : A \times Y \longrightarrow Y$ where A is a positive dimensional complex abelian variety. We have $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_C)$ in Theorem 0.1, as f has a singular fiber, and also $\rho(C) = \rho(E) = 1$, where $E = S_\eta$.

Definition 0.4. Let X be a compact Kähler manifold. We call X a *hyperkähler manifold* (HK manifold, for short) if X is simply connected and X has an everywhere non-degenerate global holomorphic 2-form σ_X s.t. $H^0(\Omega_X^2) = \mathbf{C}\sigma_X$.

Typical examples are the Hilbert schemes $S^{[n]}$ of n points on K3 surfaces S and their small deformations [Be]. When $n \geq 2$, $\rho(S^{[n]}) = \rho(S) + 1$ and $\rho(X) \leq 21$ for any small deformation of $S^{[n]}$. Note that a HK manifolds is even dimensional and both projective HK manifolds and non-projective HK manifolds are dense in the Kuranishi space.

The following theorem is due to Matsushita [M1], [M2]:

Theorem 0.5. *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers from a HK manifold of dimension $2n$ to a normal projective variety Y s.t. $0 < \dim Y < 2n$. Then, any irreducible component of the fiber is Lagrangian. In particular, any smooth fiber is a complex torus of dimension n and f is equi-dimensional. Moreover, if X is projective, then Y is a $\mathbf{Q}$-Fano variety with $\mathbf{Q}$-factorial klt singularities and $\rho(Y) = 1$. In particular, (i), (ii) as well as (iii) (as $h^1(\mathcal{O}_X) = 0$) are satisfied for an abelian fibered HK manifold.*

It is conjectured that the base space Y is always isomorphic to $\mathbf{P}^n$.

The following is one of possible generalizations of Theorem 0.1 [O2]:

Theorem 0.6. *Let $f : X \longrightarrow Y$ be an abelian fibration with properties (i), (ii), (iii). Let $\Delta = \cup_{i=1}^{k} \Delta_i \subset Y$ be the irreducible decomposition of the codimension 1 locus of the critical loci of f and let m_i be the number of prime divisors lying over Δ_i. Then $\mathrm{MW}\,(f)$ is a finitely generated abelian group of rank*

$$\mathrm{mw}\,(f) = \rho(X) - \rho(Y) - \rho(A_K) - \sum_{i=1}^{k}(m_i - 1)\ .$$

Here $\rho(A_K)$ is the rank of the Néron-Severi group of A_K, i.e., the rank of group of algebraically equivalent classes of divisors on A_K defined over K. In particular, $\rho(X) \geq 2$ and $\mathrm{mw}\,(f) \leq \rho(X) - 2$.

A similar result is also obtained independently by [Kh]. As the dual abelian variety $\hat{A}$ of A is defined over K and is isogenous to A over K, the two groups $\mathrm{MW}(f) = A(K)$ and $\mathrm{Pic}^0 A_K(K) = \hat{A}(K)$ are isomorphic modulo finite groups. This is the essential part of the proof, as it reduces the problem to the one on divisor classes on X, Y, and A_K. The rest of the proof is quite close to the proof of Theorem 0.1 [Sh] and an argument of [Ka] for certain Calabi-Yau fiber spaces. See [O2] for a complete proof.

The following is a partial generalization of Theorem 0.2:

Theorem 0.7. *For each integers $n \geq 2$ and $2 \leq \rho \leq 21$, there is an abelian fibered HK manifold $f : X \longrightarrow \mathbf{P}^n$ s.t. X is a small deformation of $S^{[n]}$ of a K3 surface S, $\rho(X) = \rho$ and $\mathrm{mw}\,(f) = \rho - 2$.*

Example 0.8. A Jacobian K3 surface $f : S \longrightarrow \mathbf{P}^1$ of Mordell-Weil rank $\rho(S) - 2$ induces an abelian fibration $f_n : S^{[n]} \longrightarrow \mathbf{P}^n$ of Mordell-Weil rank $\geq \rho(S) - 2$. For f_n, the exceptional divisor of the Hilbert-Chow morphism becomes one of two irreducible components over some critical prime divisor. Thus, from Theorem 0.6, we have $\mathrm{mw}\,(f) = \rho(S) - 2 = \rho(S^{[n]}) - 3$ and $\rho(A_K) = 1$. Note that any smooth closed fiber X_t of f_n is the product of elliptic curves. Thus $\rho(X_t) \geq 2$. In particular, $\rho(A_K) \neq \rho(X_t)$.

The crucial part of Theorem 0.7 is to compute somewhat mysterious $\rho(A_K)$:

Theorem 0.9. *Let $f : X \longrightarrow \mathbf{P}^n$ be an abelian fibered HK manifold with generic fiber A_K. Then $\rho(A_K) = 1$. In particular,* $\mathrm{mw}\,(f) = \rho(X) - 2 - \sum_{i=1}^{k}(m_i - 1)$.

For the proof, we use deformation theory. Let F be a general closed fiber of f and let $\iota : F \longrightarrow X$ be the inclusion map. As f is fibered over $\mathbf{P}^n$, by Matsushita [M3] (see also [Sa]), deformation of X that keeps fibration is of codimension 1 in the Kuranishi space. This deformation is a (part of) deformation of X that keeps F Lagrangian. Therefore, by Voisin [Vo] (an easier direction), it is of codimension $\mathrm{rank}\,\mathrm{Im}(\iota^* : H^2(X, \mathbf{Z}) \longrightarrow H^2(F, \mathbf{Z}))$. Thus $\mathrm{rank}\,\mathrm{Im}\,\iota^* = 1$. If $\rho(A_K) \geq 2$, then the specialization of divisors D_1 and D_2 on X corresponding to independent elements of $\mathrm{NS}\,(A_K)$ would yield independent elements of $\mathrm{NS}\,(F)$, a contradiction. In this way, Theorem 0.9 can be proved.

Now one can show Theorem 0.7 by starting from $f_n : S^{[n]} \longrightarrow \mathbf{P}^n$ in Example 0.8 and deforming it as in the proof for the K3 case. The argument is based on the jumping of Picard numbers under deformation [O1], again Voisin's deformation theory of Lagrangian submanifolds [Vo] (harder part), and the fact that fibered HK manifold with a bimeromorphic section over a projective base space is projective [O2]. See [O3] (which will be avaiable when this report will be published) for details.

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